Complex Analysis: Final Exam

Aletta Jacobshal 01, Wednesday 31 January 2018, 18:30–21:30
Exam duration: 3 hours

Instructions — read carefully before starting

- Write very clearly your **full name** and **student number** at the top of each answer sheet and on the envelope.
- Use the ruled paper for writing the answers and use the blank paper as scratch paper. After finishing put your answers in the envelope. **Do NOT seal the envelope!** You must return the scratch paper and the printed exam (separately from the envelope).
- Solutions should be complete and clearly present your reasoning. **When you use known results (lemmas, theorems, formulas, etc.) you must explicitly state and verify the corresponding conditions.**
- 10 points are "free". There are 6 questions and the maximum number of points is 100. The exam grade is the total number of points divided by 10.
- You are allowed to have a 2-sided A4-sized paper with handwritten notes.

Question 1 (15 points)

Evaluate

$$\text{pv}\int_{-\infty}^{\infty} \frac{e^{-ix}}{x(x^2+1)}\,\mathrm{d}x$$

using the calculus of residues.

Question 2 (15 points)

Consider the polynomial $P(z) = z^4 + \varepsilon(z-1)$ where $\varepsilon > 0$. Show that if $\varepsilon < \dfrac{r^4}{1+r}$ then the polynomial P has four zeros inside the circle $|z| = r$.

Question 3 (15 points)

Represent the function

$$f(z) = \frac{z-1}{z+1},$$

(a) (8 points) as a Taylor series around 0 and give its radius of convergence;

(b) (7 points) as a Laurent series in the domain $|z| > 1$.

Question 4 (15 points)

At which points is the function

$$f(z) = (1+i)x^2 - (1-i)y^2,$$

differentiable and at which points is it analytic? Compute the derivative of $f(z)$ at the points where it exists.

Question 5 (15 points)

Consider the function

$$f(z) = \frac{e^z}{(z-1)^2}.$$

(a) (6 points) Determine the singularities of $f(z)$ and their type (removable, pole, essential; if pole, specify the order). *Hint: You can compute the Laurent series of the function but there are also other ways to determine the type of the singularity that do not require such computation.*

(b) (6 points) Show that $f(z)$ does not have an antiderivative in $\mathbb{C} \setminus \{1\}$. *Hint: Compute the integral of f along an appropriately chosen contour.*

(c) (3 points) Explain why $f(z)$ has an antiderivative in $\mathbb{C} \setminus L$ where $L = \{x \in \mathbb{R} : x \geq 1\}$.

Question 6 (15 points)

Consider the function

$$u(x, y) = e^x \cos y,$$

(a) (7 points) Prove that the function is harmonic in $\mathbb{R}^2$.

(b) (8 points) Find a harmonic conjugate of $u(x, y)$.